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Existence and Uniqueness of Periodic Solutions for $(2n+1)$th-Order Differential Equations

Cong Fuzhong, Huang Qingdao, and Shi Shaoyun

People's Republic of China

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Applying the Schauder fixed-point theorem and the continuation theorem of Mawhin, we give the existence and uniqueness results of periodic solutions for a class of $(2n+1)$th-order nonlinear ordinary differential equations.

Key Words: $(2n+1)$th-order differential equation; periodic solution; existence and uniqueness; Schauder fixed-point theorem; continuation theorem of Mawhin.

We consider the $(2n+1)$th-order nonlinear ordinary differential equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + f(t,x) = 0, \tag{1}$$

where $f\colon R \times R \to R$ is continuous and 2π-periodic with respect to t and c_i, $i = 1, 2, \ldots, n-1$, are constants.

During the past 20 years, there has been a great amount of work in periodic solutions for the high-order Duffing equation

$$y^{(2n)} + \sum_{i=1}^{n-1} c_i y^{(2i)} + (-1)^{(n+1)} f(t,y) = 0.$$

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Among previous results [1–4], one fundamental nonresonance condition

$$\begin{aligned} 0 &\le N^{2n} + \sum_{j=1}^{n-1} (-1)^{j-n} c_j N^{2j} \\ &< \lambda \le f_y(t, y) \le \mu \\ &< (N+1)^{2n} + \sum_{j=1}^{n-1} (-1)^{j-n} c_j (N+1)^{2j}, \end{aligned}$$

where N is a nonnegative integer, is required. However, the study of odd-order differential equation is rare.

For $c_i = 0$, $i = 1, 2, \dots, n-1$, and f satisfies

$$M \ge f_x(t, x) \ge \varepsilon_0 > 0 \qquad (\text{or } -M \le f_x(t, x) \le -\varepsilon_0 < 0),$$

Li Yong and Wang Huaizhong [4] proved that (1) has a unique 2π-periodic solution.

In this note, using the Schauder fixed-point theorem and the continuation theorem of Mawhin, we give results on the existence and uniqueness of the 2π-periodic solution to (1). Assume that the following conditions hold:

(H1) f_x is continuous and there are positive constants m and M such that

$$m \le |f_x(t, x)| \le M \qquad \text{for all } (t, x). \tag{2}$$

(H2) The linear equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} = 0 \tag{3}$$

has only the constants for its 2π-periodic solutions.

(H3) $c_j \ge 0$, $j = 0, 1, \dots, n-1$, satisfies the inequality

$$\sum_{j=0}^{n-1} (2\pi)^{2(n-j)} c_j < 1. \tag{4}$$

Our results are the following.

THEOREM A. *Under conditions* (H1) *and* (H2) *above, the differential equation* (1) *has at least one* 2π*-periodic solution.*

THEOREM B. *Under conditions* (H1) *and* (H3) *above, the differential equation* (1) *has a unique* 2π*-periodic solution.*

To prove the theorems, we need the following lemma.

LEMMA 1. *Let* $x \in C^1(R)$ *and let it be a* 2π*-periodic function; and there exists* $t_0 \in R$ *such that* $x(t_0) = 0$. *Then*

$$\int_0^{2\pi} x^2(t)\,dt \le 4\pi^2 \int_0^{2\pi} [x'(t)]^2\,dt.$$

Employing the Schwarz inequality, the proof of Lemma 1 is easy. Set

$$X = \{y \in C^{2n}(R) : y(t) = y(t + 2\pi), \qquad \text{for all } t \in R\},$$

with the norm $\|\cdot\|$ defined by $\|y\| = \sum_{i=0}^{2n} \max_{[0,2\pi]} |y^{(i)}(t)|$, for each $y \in X$.

Rewrite (1) in the following form:

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + g(t,x)x = -f(t,0), \tag{5}$$

$$g(t,x) = \int_0^1 f_x(t,\theta x)\,d\theta.$$

For each $y \in X$, we consider the linear differential equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + g(t,y)x = -f(t,0). \qquad (6)_y$$

LEMMA 2. *There is* $M_1 > 0$ *such that any* 2π*-periodic solution* $x_y(t)$ *of* $(6)_y$ *satisfies*

$$\|x_y\| \le M_1, \qquad \textit{for all } y \in X.$$

Proof. Let $x_y(t)$ be any 2π-periodic solution of $(6)_y$. By Rolle's theorem, we have $\xi_j \in [0, 2\pi]$ satisfying

$$x_y^j(\xi_j) = 0, \qquad j = 1, 2, \ldots, 2n. \tag{7}$$

Applying Lemma 1,

$$\begin{aligned}
&\left(\int_0^{2\pi}\left(x_y^{(j)}\right)^2 dt\right)^{1/2} \\
&\qquad \le 2\pi\left(\int_0^{2\pi}\left(x_y^{(2j+1)}\right)^2 dt\right)^{1/2} \\
&\qquad \le (2\pi)^{2n-j}\left(\int_0^{2\pi}\left(x_y^{(2n)}\right)^2 dt\right)^{1/2}, \qquad j = 1, 2, \ldots, 2n-1. \qquad (8)
\end{aligned}$$

From (5),

$$m \le |g(t,x)| \le M, \qquad \text{for all } (t,x). \qquad (9)$$

Let $L = \sup_{t\in R}|f(t,0)|$. Take the product $(6)_y$ with $x_y(t)$ and then integrate over $[0, 2\pi]$. It follows that

$$\begin{aligned}
&\int_0^{2\pi} x_y^{(2n+1)} x_y\, dt + \sum_{i=0}^{n-1} c_i \int_0^{2\pi} x_y^{(2i+1)} x_y\, dt + \int_0^{2\pi} g(t,y)x^2\, dt \\
&\qquad = -\int_0^{2\pi} f(t,0)x_y\, dt.
\end{aligned}$$

Noting that $\int_0^{2\pi} x_y^{(2i+1)} x_y\, dt = 0$ and using (9) and the Schwarz inequality, we derive

$$\left(\int_0^{2\pi} x_y^2\, dt\right)^{1/2} \le \frac{1}{m}\left(\int_0^{2\pi}|f(t,0)|^2\, dt\right)^{1/2} \le \frac{\sqrt{2\pi}}{m}L. \qquad (10)$$

From $(6)_y$ and the Schwarz inequality, using Lemma 1, (10), and $2ab \le \varepsilon a^2 + b^2/\varepsilon$, we have

$$\begin{aligned}
\int_0^{2\pi}\left(x_y^{(2n)}\right)^2 dt &= -\int_0^{2\pi}\left(x_y^{(2n+1)}\right)\left(x_y^{(2n-1)}\right) dt \\
&= \int_0^{2\pi}\left(\sum_{i=0}^{n-1} c_i x_y^{(2i+1)} x_y^{(2n-1)}\right) dt \\
&\qquad + \int_0^{2\pi} g(t,y)x_y x_y^{(2n-1)}\, dt + \int_0^{2\pi} f(t,0)x_y^{(2n-1)}\, dt \\
&\le \sum_{i=0}^{n-1} c_i\left(\int_0^{2\pi}\left(x_y^{(2i+1)}\right)^2 dt\right)^{1/2}\left(\int_0^{2\pi}\left(x_y^{(2n-1)}\right)^2 dt\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+ M\int_0^{2\pi} |x_y x_y^{(2n-1)}|\, dt + \int_0^{2\pi} L|x_y^{(2n-1)}|\, dt \\
&\le \sum_{i=0}^{n-1} c_i (2\pi)^{2(n-i)} \int_0^{2\pi} \left(x_y^{(2n)}\right)^2 dt + \frac{1}{2} M\varepsilon \int_0^{2\pi} \left(x_y^{(2n-1)}\right)^2 dt \\
&+ \frac{1}{2\varepsilon} M \int_0^{2\pi} x_y^2\, dt + \frac{1}{2}\varepsilon \int_0^{2\pi} \left(x_y^{(2n-1)}\right)^2 dt + \frac{L^2}{2\varepsilon}\cdot 2\pi \\
&\le \left(2\pi^2 M\varepsilon + 2\pi^2\varepsilon + \sum_{i=0}^{n-1} c_i (2\pi)^{2(n-i)}\right) \int_0^{2\pi} \left(x_y^{(2n)}\right)^2 dt \\
&+ \frac{\pi L^2}{\varepsilon}\left(1 + \frac{M}{m^2}\right).
\end{aligned}
$$

Choose $\varepsilon > 0$ so that

$$
1 - \sum_{j=0}^{n-1} (2\pi)^{2(n-j)} c_j - 2\pi^2\varepsilon - 2\pi^2 M\varepsilon \triangleq A > 0;
$$

then we get

$$
\int_0^{2\pi} \left(x_y^{(2n)}\right)^2 dt \le \frac{\pi L^2}{A\varepsilon}\left(1 + \frac{M}{m^2}\right) \triangleq B^2 L^2. \tag{11}
$$

On the basis of (7), (8), (11), and the Schwarz inequality, we obtain

$$
\begin{aligned}
\left| x_y^{(j)}(t) \right| &= \left| \int_{\xi_j}^{t} x_y^{(j+1)}\, dt \right| \\
&\le (2\pi)^{1/2} \left(\int_0^{2\pi} \left(x_y^{(j+1)}\right)^2 dt \right)^{1/2} \\
&\le (2\pi)^{2n-j+1/2} BL, \\
&\text{for all } t \in [0, 2\pi], \quad j = 1, 2, \ldots, 2n-1. \qquad (12)
\end{aligned}
$$

Also by $(6)_y$,

$$
\int_0^{2\pi} g(t, y) x_y\, dt = -\int_0^{2\pi} f(t, 0)\, dt.
$$

Using (9), there exists $\xi \in [0, 2\pi]$ satisfying

$$\left| x_y(\xi) \right| \le \frac{1}{2\pi m} \int_0^{2\pi} |f(t,0)|\, dt \le \frac{L}{m}. \tag{13}$$

From (8), (11), and (13),

$$\begin{aligned} \left| x_y(t) \right| &\le \left| x_y(\xi) \right| + \left| \int_\xi^t x_y(t)\, dt \right| \\ &\le \frac{L}{m} + (2\pi)^{2n-1/2} BL \triangleq LD, \qquad \text{for all } t \in [0, 2\pi]. \end{aligned} \tag{14}$$

According to $(6)_y$, for all $t \in [0, 2\pi]$,

$$\begin{aligned} \left| x_y^{(2n)}(t) \right| &= \left| \int_{\xi_{2n}}^t x_y^{(2n+1)}(t)\, dt \right| \\ &\le \sum_{i=0}^{n-1} c_i \int_0^{2\pi} |x_y^{(2i+1)}|\, dt + \int_0^{2\pi} |g(t,y)||x_y|\, dt \\ &\quad + \int_0^{2\pi} |f(t,0)|\, dt. \end{aligned} \tag{15}$$

Using (12), (14), and (15), there is constant $E > 0$, such that

$$\left| x_y^{(2n)}(t) \right| \le EL, \qquad \text{for all } t \in [0, 2\pi],$$

which, with (12) and (14), completes the proof of Lemma 2.

THEOREM C. *Assume that* $h(t) = h(t + 2\pi)$ *is continuous and satisfies*

$$m \le |h(t)| \le M \qquad \textit{for all } t \in [0, 2\pi],$$

where m, M *are constants, and* c_i, $i = 0, 1, \dots, n-1$, *satisfies the inequality* (4). *Then the equation*

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + h(t) x = 0 \tag{16}$$

has only a zero solution.

The proof of Theorem C is similar to that of Lemma 2. We only need to note that in (12), (14), and (15), $L = 0$, THEOREM D. *Assume that* $h(t)$, c_i, $i = 0, 1, \ldots, n-1$, *satisfies the conditions in Theorem C. Then the linear differential equation*

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + h(t)x = e(t), \tag{17}$$

where $e(t) = e(t + 2\pi)$ *is continuous, has a unique* 2π*-periodic solution.*

Proof. By Theorem C, (16) has only a zero solution. Since the equation (17) is linear, the uniqueness implies the existence. The proof is completed.

Proof of Theorem B. First prove the uniqueness. Let $x_1(t)$ and $x_2(t)$ be any two 2π-periodic solution of (1). Then $x(t) = x_1(t) - x_2(t)$ is a 2π-periodic solution of the following equation:

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + \int_0^1 f_x(t, x_2(t) + \theta x(t))\,d\theta\, x = 0.$$

From the hypothesis (H1), we see

$$m \leq \left| \int_0^1 f_x(t, x_2(t) + \theta x(t))\,d\theta \right| \leq M.$$

Hence by Theorem C, $x(t) \equiv 0$, on R.

We next prove the existence. For each $y \in X$, define

$$Py = x_y,$$

where x_y is a 2π-periodic solution of $(6)_y$. By Theorem D, P is well defined.

Let x_n, $x \in X$ such that $\|x_n - x\| \to 0$, as $n \to \infty$. We assert: $\|Px_n - Px\| \to 0 (n \to \infty)$. If not, by Lemma 2, $\{Px_n\}$ would have a convergent subsequence, still denoting it by $\{Px_n\}$, such that $\|Px_n - y\| \to 0 (n \to \infty)$ and $Px \neq y$. Set $u_n = Px_n - Px$. Then $u_n(t)$ satisfies

$$u_n^{(2n+1)} + \sum_{i=0}^{n-1} c_i u_n^{(2i+1)} + g(t, x_n)u_n + \big(g(t, x_n) - g(t, x)\big)Px = 0,$$

and u_n is 2π-periodic. Hence by the continuity of g, as $n \to \infty$, we get

$$u_0^{(2n+1)} + \sum_{i=0}^{n-1} c_i u_0^{(2i+1)} + g(t, x)u_0 = 0, \tag{18}$$

where $u_0(t) = y(t) - Px(t)$. By Theorem C, (18) has only the zero solution, which contradicts $y \neq Px$. By (18) and the Arzela–Ascoli theorem, it follows that P is completely continuous. From Lemma 2, PX is bounded. Applying the Schauder fixed-point theorem, P has a fixed point $x(t)$ in X, which is a solution of (1). This completes the proof of Theorem B.

Now we prove Theorem A by applying the continuation theorem of Mawhin (see, e.g., [5] or [6]).

Consider the following auxiliary equations:

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + \lambda g(t,x)x = -\lambda f(t,0), \qquad 0 < \lambda \leq 1, \quad (19)_\lambda$$

using the notation in (5). Define the linear operator L: $C_{2\pi}^{2n} \to L^2(0, 2\pi)$ by

$$Lx = x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)}, \qquad x \in C_{2\pi}^{2n}.$$

By (2) in Condition (H1), the operator $x(\cdot) \to f(\cdot, x(\cdot))$ maps L^2 into itself.

Multiplying each side of $(19)_\lambda$ by $x(t)$, integrating over $[0, 2\pi]$, and using the boundary conditions, we get

$$\lambda \int_0^{2\pi} g(t, x(t)) x^2(t)\, dt = -\lambda \int_0^{2\pi} f(t,0) x(t)\, dt.$$

Dividing out λ and using $|g(t,x)| \geq m$ for all (t,x), we get

$$\int_0^{2\pi} x^2(t)\, dt \leq M_0 \tag{20}$$

for a constant M_0 independent of x and λ. This gives an *a priori* bound in L^2.

Instead of working in L^2, one could work in $C_{2\pi}$, using (20) and the equation $(19)_\lambda$ to obtain a bound in $C_{2\pi}$.

Clearly, according to (H1) and the definition of topological degree, the mapping from the constants (the kernel of the operator L) to the constants, given by

$$\bar{x} \to \frac{1}{2\pi} \int_0^{2\pi} f(t, \bar{x})\, dt, \qquad \bar{x} \in R,$$

has nonzero topological degree on a large interval containing 0.

Employing the continuation theorem of Mawhin, we can complete the proof of Theorem A.

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